# Computer vision: models, learning and inference

Chapter 2

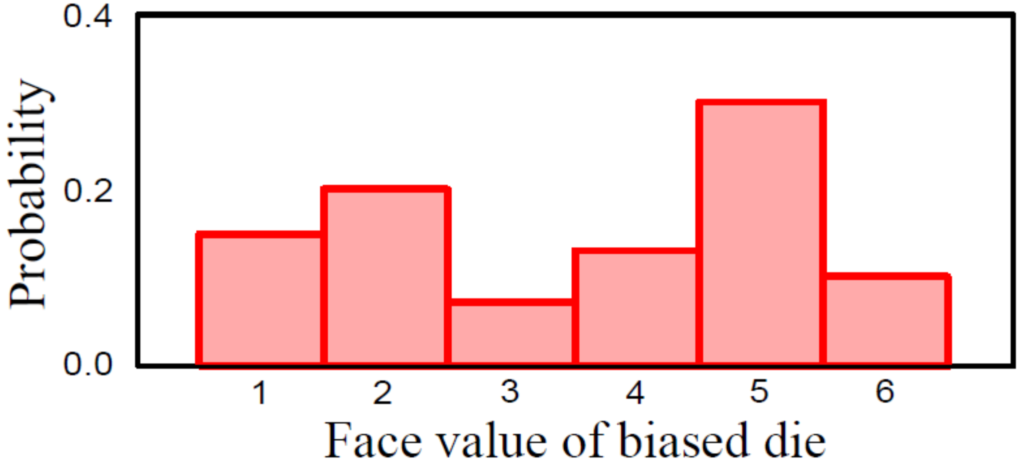
Introduction to probability

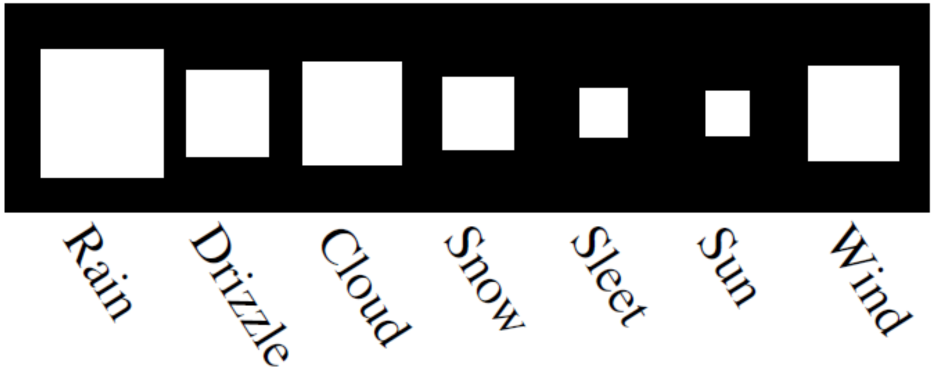
Please send errata to s.prince@cs.ucl.ac.uk

# Random variables

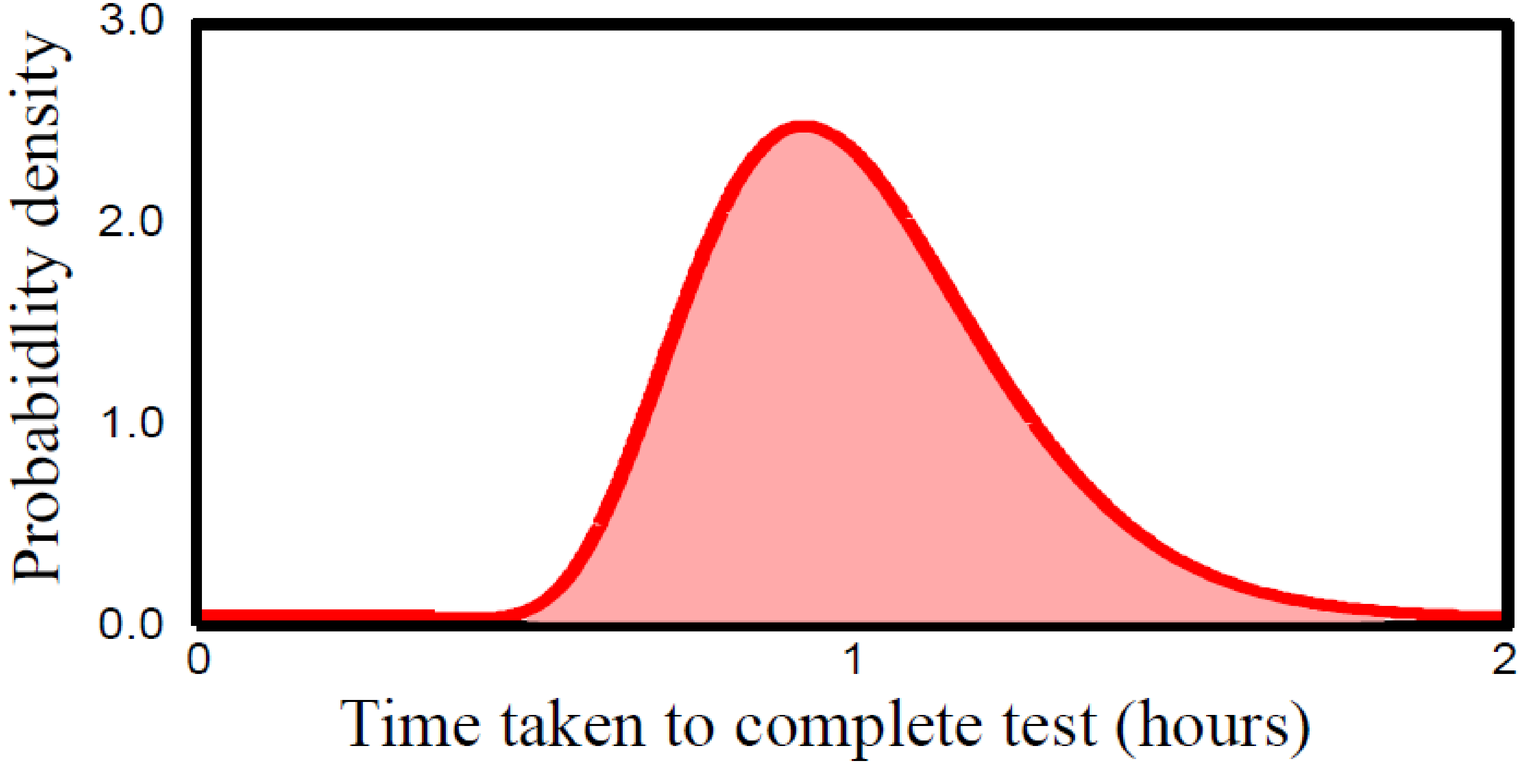
- A random variable $x$ denotes a quantity that is uncertain
- May be result of experiment (flipping a coin) or a real world measurements (measuring temperature)
- If observe several instances of $x$ we get different values
- Some values occur more than others and this information is captured by a probability distribution

# Discrete Random Variables



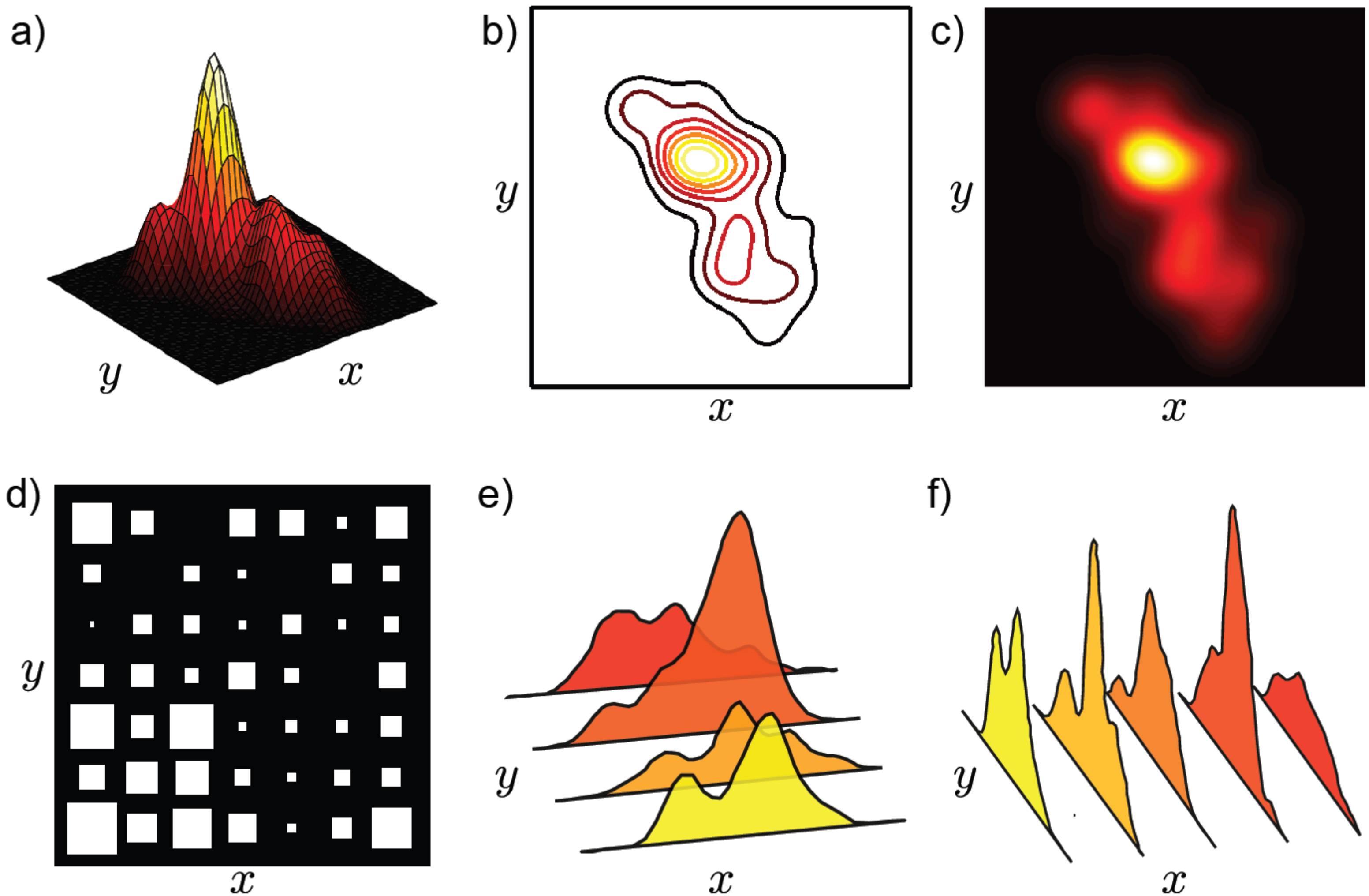
# Continuous Random Variable



# Joint Probability

- Consider two random variables $x$ and $y$
- If we observe multiple paired instances, then some combinations of outcomes are more likely than others
- This is captured in the joint probability distribution
- Written as $Pr(x,y)$
- Can read $Pr(x,y)$ as "probability of $x$ and $y$"

# Joint Probability

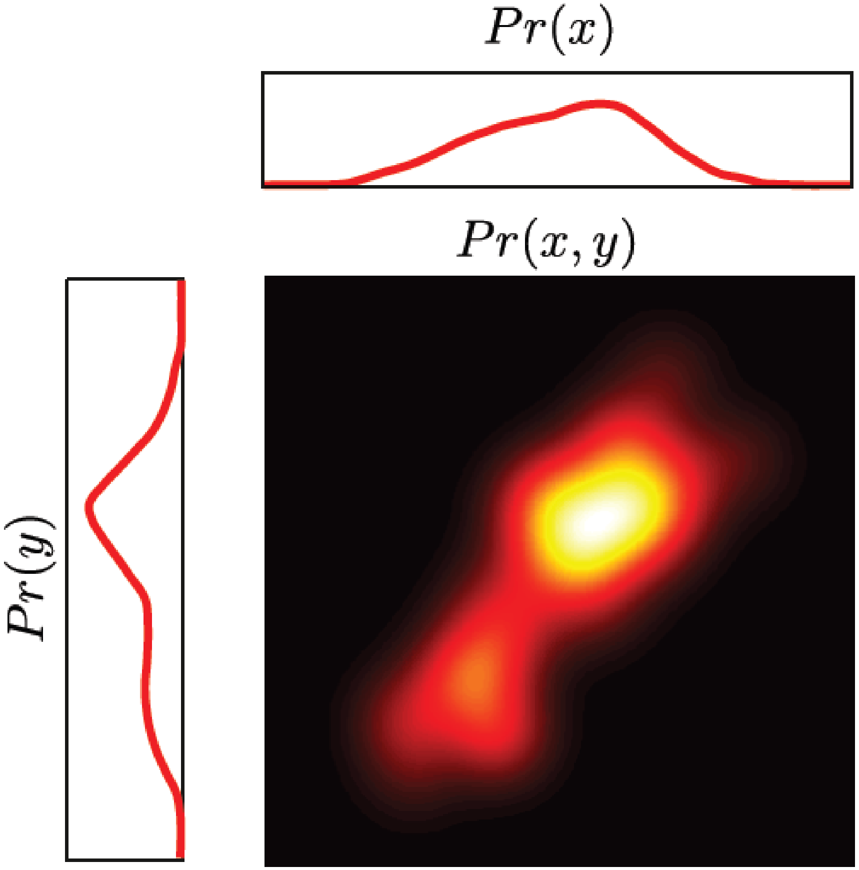


# Marginalization

We can recover probability distribution of any variable in a joint distribution by integrating (or summing) over the other variables

$$Pr(x) = \int Pr(x, y)\, dy$$

$$Pr(y) = \int Pr(x, y)\, dx$$

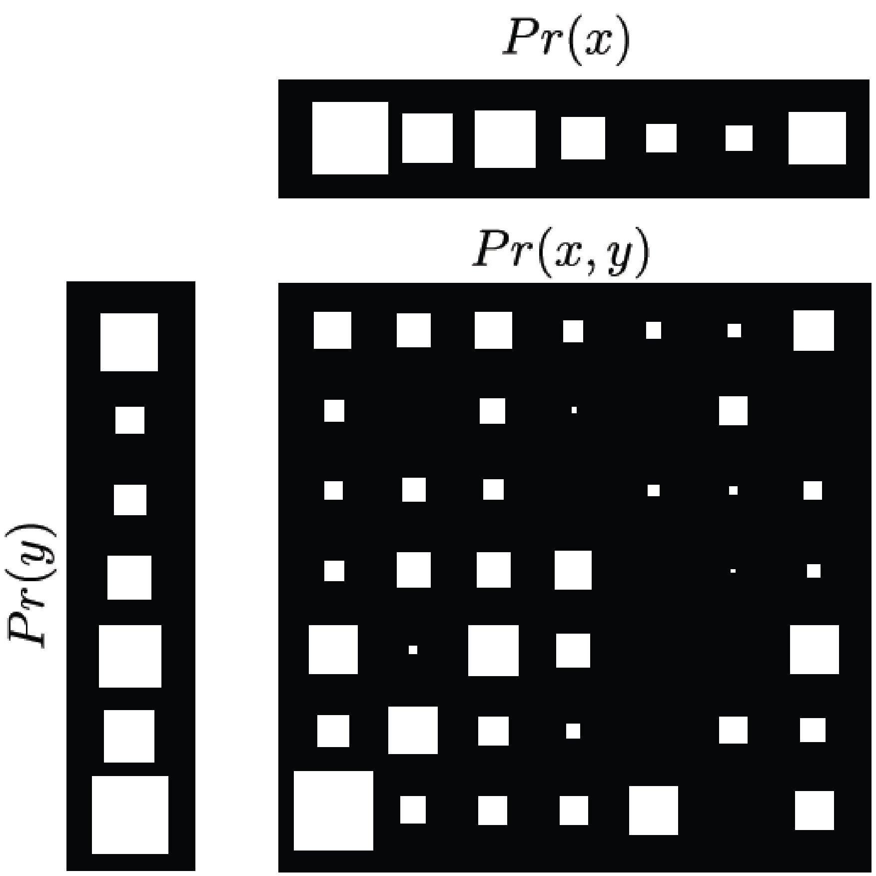


# Marginalization

We can recover probability distribution of any variable in a joint distribution by integrating (or summing) over the other variables

$$Pr(x) = \sum_{y} Pr(x, y)$$

$$Pr(y) = \sum_{x} Pr(x, y)$$

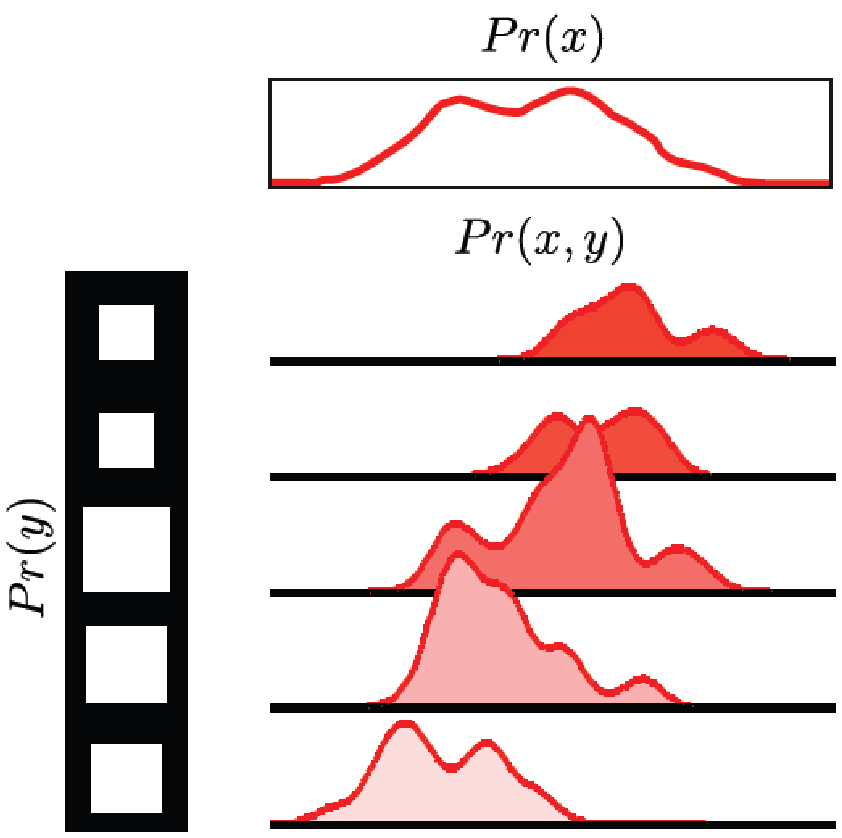


# Marginalization

We can recover probability distribution of any variable in a joint distribution by integrating (or summing) over the other variables

$$Pr(x) = \sum_y Pr(x,y)$$

$$Pr(y) = \int Pr(x,y)\, dx$$



# Marginalization

We can recover probability distribution of any variable in a joint distribution by integrating (or summing) over the other variables

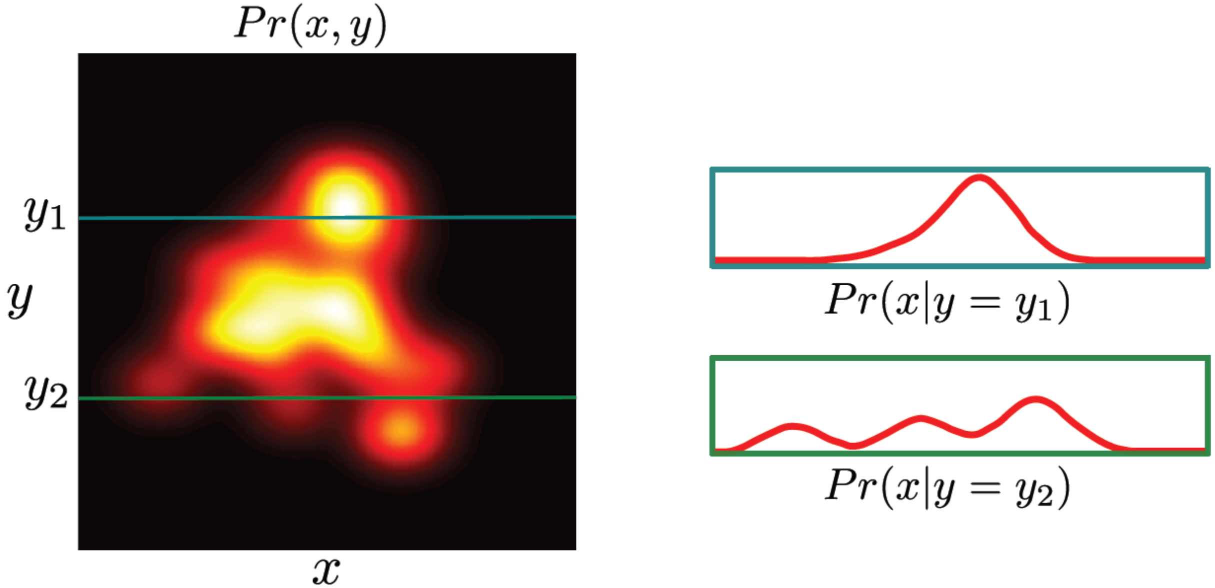
$$
\begin{aligned}
Pr(x) &= \int Pr(x,y)\, dy \\
Pr(y) &= \int Pr(x,y)\, dx
\end{aligned}
$$

Works in higher dimensions as well – leaves joint distribution between whatever variables are left

$$Pr(x,y) = \sum_{w} \int Pr(w,x,y,z)\, dz$$

# Conditional Probability

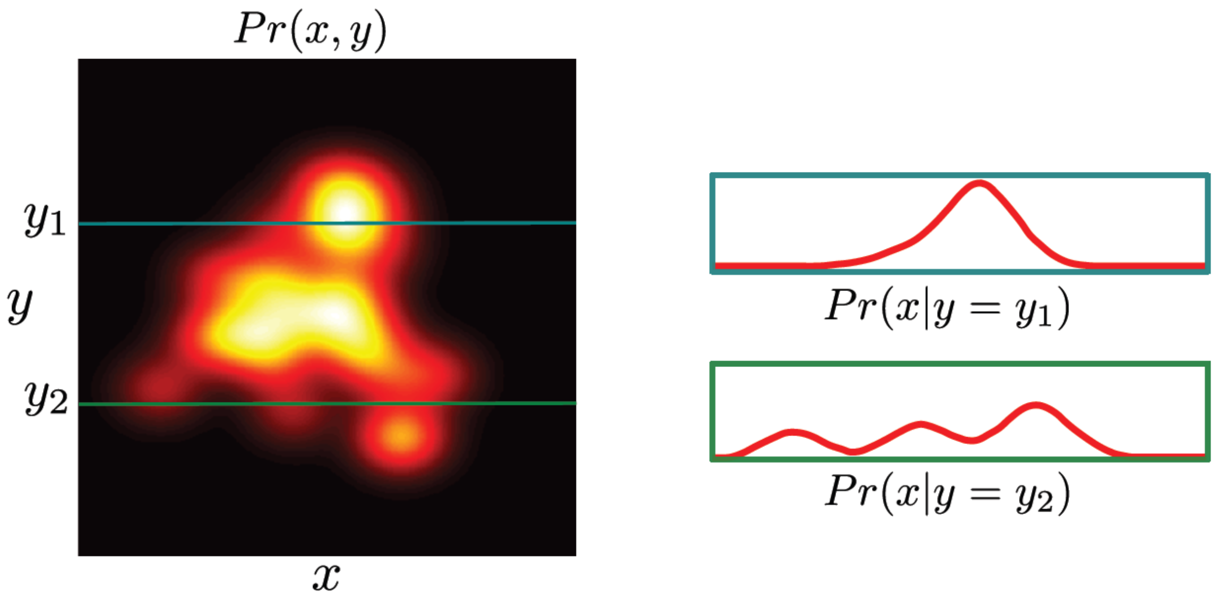
- Conditional probability of $x$ given that $y=y_1$ is relative propensity of variable $x$ to take different outcomes given that y is fixed to be equal to $y_1$.
- Written as $Pr(x|y=y_1)$



# Conditional Probability

- Conditional probability can be extracted from joint probability
- Extract appropriate slice and normalize

$$Pr(x|y=y^*) = \frac{Pr(x,y=y^*)}{\int Pr(x,y=y^*)dx} = \frac{Pr(x,y=y^*)}{Pr(y=y^*)}$$



# Conditional Probability

$$Pr(x|y=y^*) = \frac{Pr(x,y=y^*)}{\int Pr(x,y=y^*)dx} = \frac{Pr(x,y=y^*)}{Pr(y=y^*)}$$

- More usually written in compact form

$$Pr(x|y) = \frac{Pr(x,y)}{Pr(y)}$$

- Can be re-arranged to give

$$Pr(x,y) = Pr(x|y)Pr(y)$$

$$Pr(x,y) = Pr(y|x)Pr(x)$$

# Conditional Probability

$$Pr(x,y) = Pr(x|y)Pr(y)$$

- This idea can be extended to more than two variables

$$\begin{aligned} Pr(w,x,y,z) &= Pr(w,x,y|z)Pr(z) \\ &= Pr(w,x|y,z)Pr(y|z)Pr(z) \\ &= Pr(w|x,y,z)Pr(x|y,z)Pr(y|z)Pr(z) \end{aligned}$$

# Bayes' Rule

From before:

$$Pr(x,y) = Pr(x|y)Pr(y)$$

$$Pr(x,y) = Pr(y|x)Pr(x)$$

Combining:

$$Pr(y|x)Pr(x) = Pr(x|y)Pr(y)$$

Re-arranging:

$$\begin{aligned} Pr(y|x) &= \frac{Pr(x|y)Pr(y)}{Pr(x)} \\ &= \frac{Pr(x|y)Pr(y)}{\int Pr(x,y)\, dy} \\ &= \frac{Pr(x|y)Pr(y)}{\int Pr(x|y)Pr(y)\, dy} \end{aligned}$$

# Bayes' Rule Terminology

Likelihood – propensity for observing a certain value of x given a certain value of y

Prior – what we know about y before seeing x

$$Pr(y|x) = \frac{Pr(x|y)Pr(y)}{\int Pr(x|y)Pr(y)\,dy}$$

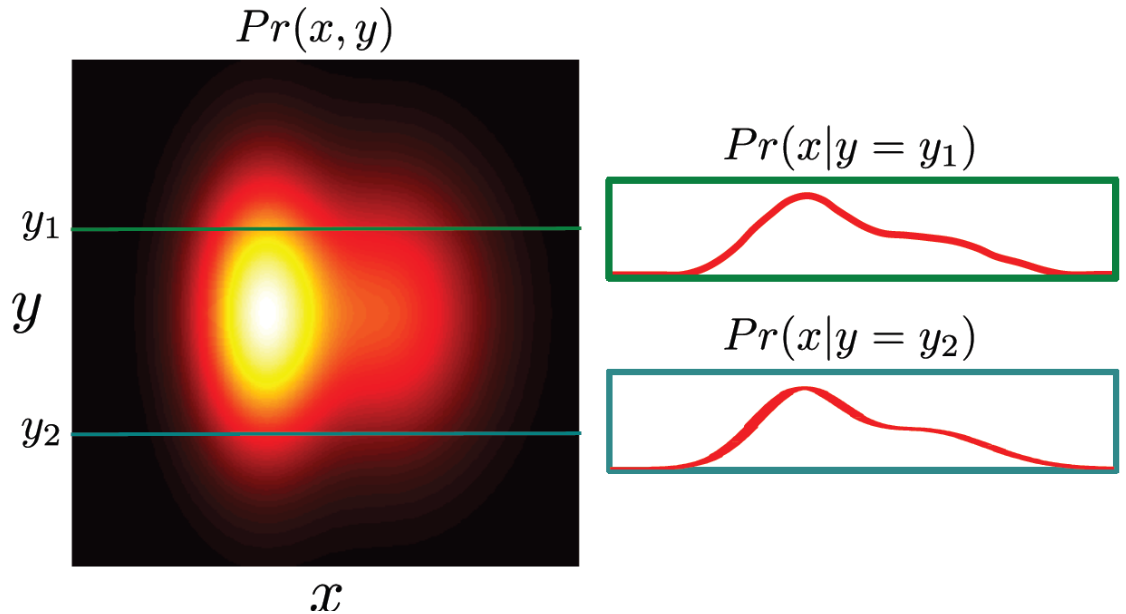
Posterior – what we know about y after seeing x

Evidence –a constant to ensure that the left hand side is a valid distribution

# Independence

- If two variables x and y are independent then variable x tells us nothing about variable y (and vice-versa)

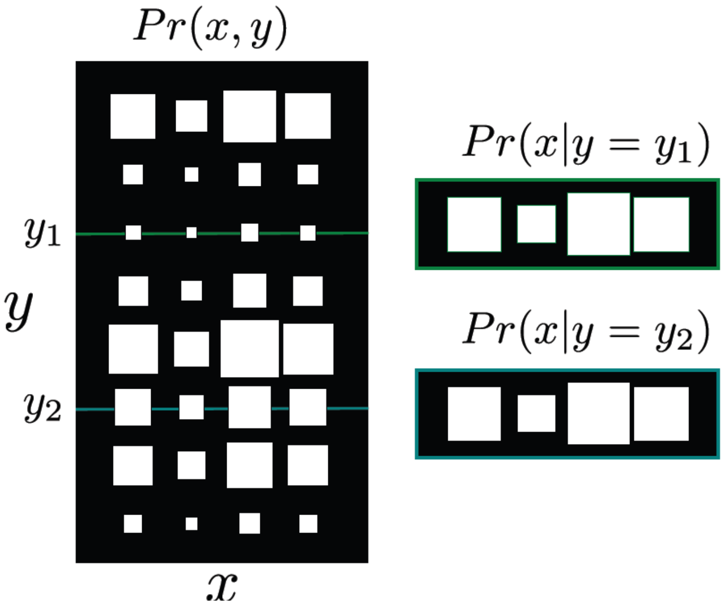
$$Pr(x|y) = Pr(x)$$
$$Pr(y|x) = Pr(y)$$



# Independence

- If two variables x and y are independent then variable x tells us nothing about variable y (and vice-versa)

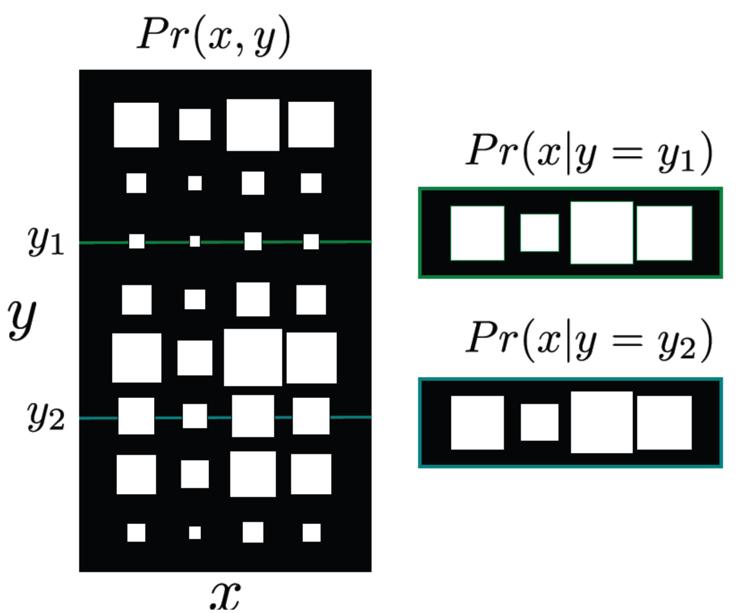
$$Pr(x|y) = Pr(x)$$
$$Pr(y|x) = Pr(y)$$



# Independence

- When variables are independent, the joint factorizes into a product of the marginals:

$$
\begin{aligned}
Pr(x,y) &= Pr(x|y)Pr(y) \\
&= Pr(x)Pr(y)
\end{aligned}
$$



# Expectation

Expectation tell us the expected or average value of some function $\mathrm{f}[\mathrm{x}]$ taking into account the distribution of $\mathrm{x}$

Definition:

$$\mathrm{E}[f[x]] = \sum_x f[x]Pr(x)$$

$$\mathrm{E}[f[x]] = \int f[x]Pr(x)\, dx$$

# Expectation

Expectation tell us the expected or average value of some function $\mathrm{f}[\mathrm{x}]$ taking into account the distribution of $\mathrm{x}$

Definition in two dimensions:

$$\mathrm{E}[f[x,y]] = \iint f[x,y]Pr(x,y)\,dx\,dy$$

# Expectation: Common Cases

$$\mathrm{E}[f[x]] = \int f[x]Pr(x)\,dx$$

| Function $f[\bullet]$ | Expectation |
|---|---|
| $x$ | mean, $\mu_x$ |
| $x^k$ | $k^{th}$ moment about zero |
| $(x-\mu_x)^k$ | $k^{th}$ moment about the mean |
| $(x-\mu_x)^2$ | variance |
| $(x-\mu_x)^3$ | skew |
| $(x-\mu_x)^4$ | kurtosis |
| $(x-\mu_x)(y-\mu_y)$ | covariance of $x$ and $y$ |

# Expectation: Rules

$$E[f[X]] = \int f[x] Pr(X = x) dx$$

Rule 1:

Expected value of a constant is the constant

$$\mathrm{E}[\kappa] = \kappa$$

# Expectation: Rules

$$E[f[X]] \quad = \quad \int f[x]Pr(X=x)dx$$

Rule 2:

Expected value of constant times function is constant times expected value of function

$$\mathrm{E}[kf[x]] = k\mathrm{E}[f[x]]$$

# Expectation: Rules

$$E[f[X]] \quad = \quad \int f[x]Pr(X=x)dx$$

Rule 3:

Expectation of sum of functions is sum of expectation of functions

$$\mathrm{E}[f[x]+g[x]] = \mathrm{E}[f[x]] + \mathrm{E}[g[x]]$$

# Expectation: Rules

$$E[f[X]] = \int f[x] Pr(X = x) dx$$

Rule 4:

Expectation of product of functions in variables x and y
is product of expectations of functions if x and y are independent

$$\mathrm{E}[f[x]g[y]] = \mathrm{E}[f[x]]\mathrm{E}[g[y]] \qquad \text{if } x, y \text{ independent}$$

# Conclusions

- Rules of probability are compact and simple
- Concepts of marginalization, joint and conditional probability, Bayes rule and expectation underpin all of the models in this book
- One remaining concept – conditional expectation – discussed later